Math 199 CD2: Limit and ϵ-δ

August 31, 2021

1. Let $E(h) = h^3$. We want to show $\lim\limits_{h \to 0} E(h) = 0$.

   (a) If $\epsilon = 1$, then find δ so that when $0 < |h| < \delta$, we know $|E(h)| < \epsilon$.

   $|E(h)| < 1 \Leftrightarrow |h^3| < 1 \Leftrightarrow -1 < h^3 < 1 \Leftrightarrow -1 < h < 1$

   $\Leftrightarrow 0 < |h| < 1$

   $\Rightarrow \delta = 1$

   (b) If $\epsilon = \frac{1}{8}$, then find δ so that when $0 < |h| < \delta$, we know $|E(h)| < \epsilon$.

   $|h^3| < \frac{1}{8} \Leftrightarrow \frac{-1}{8} < h^3 < \frac{1}{8} \Leftrightarrow \frac{-1}{2} < h < \frac{1}{2} \Leftrightarrow 0 < |h| < \frac{1}{2}$

   $\Rightarrow \delta = 1/2$

   (c) If we ϵ is not given explicitly, find δ in terms of ϵ so that when $0 < |h| < \delta$, we know $|E(h)| < \epsilon$.

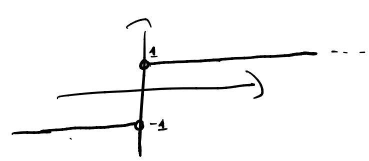
   $|h^3| < \varepsilon \Leftrightarrow -\varepsilon < h^3 < \varepsilon \Leftrightarrow -\sqrt[3]{\varepsilon} < h < \sqrt[3]{\varepsilon}$

   $\Rightarrow 0 < |h| < \sqrt[3]{\varepsilon}$

   $\Rightarrow \delta = \sqrt[3]{\varepsilon}$

2. For each of the following, use the ϵ-δ definition of the limit to show that the limit does not exist. Use words!

   (a) $\lim\limits_{x \to 0} \frac{|x|}{x}$. Plot out the function

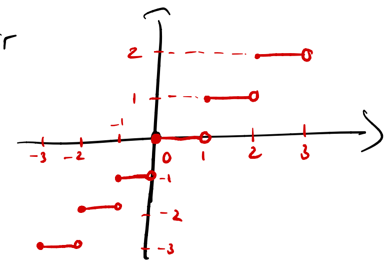


   let $\varepsilon = 0.5$ can you find any δ to create a box

   In fact you can pick any $\varepsilon < 1$

(b) $\lim_{x\to 1} \lfloor x \rfloor$ where $\lfloor x \rfloor$ is x rounded down to the nearest integer. For example, $\lfloor 1.7 \rfloor = \lfloor 1.2 \rfloor = 1$, $\lfloor -1/2 \rfloor = \lfloor -2/3 \rfloor = -1$

Function looks sth like a stair



Pick $\varepsilon = 0.5$ again, can you find the box?

3. We want to show that $\lim_{x\to 2}(2-3x) = -4$.

(a) Fill in the blanks to set up the problem using a limit of zero at zero.
Let $E(h) = (2-3(2+h)) - (-4) = -3h$. We say that $\underline{E(h)}$ has limit $\underline{0}$ at $\underline{0}$ if
for every challenge number $\epsilon > 0$,
there is a response number $\delta > 0$ such that
if the input $\underline{h}$ is strictly between $\underline{-\delta}$ and $\underline{\delta}$, but $\underline{h}$ is not equal to $\underline{0}$,
then the output $\underline{E(h)}$ will be strictly between $\underline{-\varepsilon}$ and $\underline{\varepsilon}$.

(b) Fill in the blanks to set up the problem using the traditional definition of the limit.
We say that $\underline{f(x)=2-3x}$ has limit $\underline{-4}$ at $\underline{2}$ if
for every challenge number $\epsilon > 0$,
there is a response number $\delta > 0$ such that
if $0 < \underline{|x-2|} < \delta$,
then $\underline{|f(x)+4|} < \epsilon$.

(c) How are these two limit definitions the same? How are they different? Discuss with your group.

(d) Use either method to show $\lim_{x\to 2}(2-3x) = -4$.

Apply argument

$|f(x)+4| < \varepsilon$

$\iff |2-3x+4| < \varepsilon$

$\iff |-3x+6| < \varepsilon \iff -\varepsilon < -3x+6 < \varepsilon$

$\iff -\varepsilon < -3(x-2) < \varepsilon$

$\iff \frac{\varepsilon}{3} < x-2 < \frac{\varepsilon}{-3}$

$\delta = \frac{-\varepsilon}{3}$

$\iff |x-2| < \frac{\varepsilon}{3}$

4. Suppose $\lim_{x \to 0} E(x) = 0$. Use the ϵ-δ definition of a limit to prove the following:

(a) $\lim_{x \to 0} 2E(x) = 0$

$\lim_{x \to 0} E(x) = 0$ then for all $\varepsilon > 0$, exists $\delta > 0$ s.t. if $|x| < \delta$ then $|E(x)| < \varepsilon$

$\Rightarrow$ For any $\varepsilon' = \frac{\varepsilon}{2} > 0$, exists $\delta > 0$ s.t. $|x| < \delta$ then $|E(x)| < \varepsilon' = \frac{\varepsilon}{2} \Leftrightarrow 2|E(x)| < \varepsilon'$ ∎