Math 199 CD2: Asymptotics and Limit at Infinity

September 2, 2021

1. Show that the following equation has at least 1 solution using intermediate value theorem:

   (a) $x^3 + x + 1 = 0$

   (b) $x^5 + x^2 + 1 = 0$

2. Calculate $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$.

3. In each part below, invent a function $f(x)$ with the desired properties, or show no such function can exist.

   (a) $\lim_{x\to\infty} f(x) - x = \infty$ and $\lim_{x\to\infty} 2x - f(x) = \infty$. Hint: Think of function in the form $f(x) = cx$ where c is a constant

   (b) $\lim_{x\to\infty} f(x) - x = 2$ and $\lim_{x\to\infty} 2x - f(x) = 2$. Hint: Limit summation might be helpful here

   (c) $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} e^x f(x) = \infty$.

   (d) $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} \dfrac{f(x)}{\ln(x)} = 0$.

4. **Exponentials are faster than polynomails.** In this problem you will prove that (growing) exponential functions grow faster than polynomials, a fact that you can cite later and will be very useful.

(a) Expand $(x+y)^4$. Recall 'binomial theorem':

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n}x^0 y^n = \sum_{i=1}^{n}\binom{n}{i}x^i y^{n-i}$$

where:

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

(b) Use the binomial theorem to show that if $\alpha \geq 0$, then $(1+\alpha)^n \geq 1+n\alpha+\frac{n(n-1)}{2}\alpha^2$.

(c) Calculate $\lim_{n\to\infty}\frac{n}{(1+\alpha)^n}$ using (b). I'm looking for the "squeez"

(d) Show that $\lim_{x\to\infty}\frac{x^2}{2^x} = 0$ using (c) and the transformation

$$\lim_{x\to\infty}\frac{x^2}{2^x} = \left(\lim_{x\to\infty}\frac{x}{(\sqrt{2})^x}\right)^2$$

(e) Show that $\lim_{x\to\infty} \frac{x^a}{c^x} = 0$ for any $a \geq 0$ and $c > 1$.

5. **Computing more Limits**

(a)

$$\lim_{x\to\infty} \frac{x^3-2}{3x^2+4x-1}$$

(b)

$$\lim_{x\to\infty} \frac{2x^2-x+1}{4x^2-3x-1}$$

(c)

$$\lim_{x\to\infty} \frac{2x^2-1}{4x^3-5x-1}$$

(d) $e^{-3x}\cos x$

6. **Little-o Notation.** The following notation is not taught in this course, but it is essential for any engineer or computer scientist. We say that $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ (in this sense, f grows more slowly than g). The '=' used here is not a true equality, as many distinct functions can be $o(g(x))$. Confirm the following.You don't need to finish the whole problem but it's a good practice.

   (a) $x = o(x^2)$ and $x^{3/2} + \sqrt{x} = o(x^2)$.

   (b) For any $\alpha, \beta > 0$, $x^\alpha = o(x^{\alpha+\beta})$.

   (c) For any $a \geq 0$ and $c > 1$, $x^a = o(c^x)$.

   (d) $\ln(x) = o(x)$. Use the previous part but it's a bit tricky! A useful identity that you will use a lot is $x = e^{\ln(x)}$. Make it a fun exercise to verify this identity but feel free to just use it for now in this problem

   (e) For any $\alpha > 0$, $\ln(x) = o(x^\alpha)$. (Use a clever change of variables.)

   (f) $2^x = o(3^x)$.

   (g) For any $c > d > 1$, $d^x = o(c^x)$.

   (h) $\ln(\ln(x)) = o(\ln(x))$.

   (i) $e^{\sqrt{\ln(x)}} = o(\sqrt{x})$.

   (j) For any $\alpha > 0$, $e^{\sqrt{\ln(x)}} = o(x^\alpha)$.

   (k) $\ln(x) = o(e^{\sqrt{\ln(x)}})$

   (l) $c^x = o(x^x)$.

Conclusions: logarithms are slower than polynomials, which are slower than exponentials. But there are still functions slower, faster, and in between.