Math 199 CD2: Asymptotics and Limit at Infinity

September 2, 2021

1. Show that the following equation has at least 1 solution using intermediate value theorem:

   (a) $x^3 + x + 1 = 0$

   $f(x) = x^3 + x + 1$

   $f(2) = 8 + 2 + 1 = 11$

   $f(-2) = -8 - 2 + 1 = -10$

   $\Rightarrow -10 < f(x) < 10$

   $\Rightarrow$ Exists some c between -2 & 2 s.t. $f(c) = 0$

   (b) $x^5 + x^2 + 1 = 0$

   2 & -2 would work again

2. Calculate $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$.

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$-x \leq x \sin\left(\frac{1}{x}\right) \leq x$$

$$0 = \lim_{x \to 0} -x \leq \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) \leq \lim_{x \to 0} x = 0$$

$$\Downarrow$$

$$= 0.$$

3. In each part below, invent a function $f(x)$ with the desired properties, or show no such function can exist.

(a) $\lim_{x\to\infty} f(x) - x = \infty$ and $\lim_{x\to\infty} 2x - f(x) = \infty$. Hint: Think of function in the form $f(x) = cx$ where c is a constant

$$f(x) = \frac{3}{2}x$$

(b) $\lim_{x\to\infty} f(x) - x = 2$ and $\lim_{x\to\infty} 2x - f(x) = 2$. Hint: Limit summation might be helpful here

$$\lim_{x\to\infty} \left[(f(x) - x) + (2x - f(x)) \right] = \lim_{x\to\infty} x = \infty$$

But then from hypothesis $\Rightarrow$ Can't exist such function.

$$\lim_{x\to\infty} \left[(f(x) - x) + (2x - f(x)) \right] = 4$$

(c) $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} e^x f(x) = \infty$.

$e^{-\frac{1}{2}x}$ or $\frac{1}{x}$ would work (Check)

(d) $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} \dfrac{f(x)}{\ln(x)} = 0$.

$$f(x) = \sqrt{\ln(x)}$$

4. **Exponentials are faster than polynomails.** In this problem you will prove that (growing) exponential functions grow faster than polynomials, a fact that you can cite later and will be very useful.

(a) Expand $(x+y)^4$. Recall 'binomial theorem':

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n}x^0 y^n = \sum_{i=1}^{n}\binom{n}{i}x^i y^{n-i}$$

where:

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

A trick to find small coefficient !!

$(x+y)^0 \to 1$
$(x+y)^1 \to 1 \quad 1$
$(x+y)^2 \to 1 \quad 2 \quad 1$
$(x+y)^3 \to 1 \quad 3 \quad 3 \quad 1$
$1 \quad 4 \quad 6 \quad 4 \quad 1$

$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

(b) Use the binomial theorem to show that if $\alpha \geq 0$, then $(1+\alpha)^n \geq 1+n\alpha+\frac{n(n-1)}{2}\alpha^2$.

$$(1+\alpha)^n = 1 + \underbrace{\binom{n}{1}\alpha + \binom{n}{2}\alpha^2 + \binom{n}{3}\alpha^3 + \cdots}$$

$$\geq 1 + n\alpha + \frac{n(n-1)\alpha^2}{2} \quad (\text{why ?})$$

(c) Calculate $\lim_{n\to\infty}\frac{n}{(1+\alpha)^n}$ using (b). I'm looking for the "squeez"

$$0 \leq \lim_{n\to\infty}\frac{n}{(1+\alpha)^n} \leq \lim_{n\to\infty}\frac{n}{1+n\alpha+\frac{n(n-1)}{2}\alpha^2} = 0$$

$\Rightarrow \ldots$

(d) Show that $\lim_{x\to\infty}\frac{x^2}{2^x} = 0$ using (c) and the transformation

$$\lim_{x\to\infty}\frac{x^2}{2^x} = \Big(\underbrace{\lim_{x\to\infty}\frac{x}{(\sqrt{2})^x}}\Big)^2$$

since $\sqrt{2} > 1$
We have this limit is 0
by part (c).

(e) Show that $\lim_{x\to\infty} \frac{x^a}{c^x} = 0$ for any $a \geq 0$ and $c > 1$.

$a = 0$ is easy since $\lim_{x\to\infty} \frac{1}{c^x} = 0$

On the other hand

$a > 0 \Rightarrow \lim_{x\to\infty} \frac{x^a}{c^x} = \lim_{x\to\infty} \left(\frac{x}{(c^{1/a})^x}\right)^a = 0$ by (c)

$c > 1$ & $c^{1/a} > 1$ (verify this)

5. **Computing more Limits**

(a)

$$\lim_{x\to\infty} \frac{x^3 - 2}{3x^2 + 4x - 1} = \lim_{x\to\infty} \frac{1 - \frac{2}{x^3}}{\frac{3}{x} + \frac{4}{x^2} - \frac{1}{x^3}} = \infty$$

$\to 1$

$\to 0$

(b)

$$\lim_{x\to\infty} \frac{2x^2 - x + 1}{4x^2 - 3x - 1} = \lim_{x\to\infty} \frac{2 - \frac{1}{x} + \frac{1}{x^2}}{4 - \frac{3}{x} - \frac{1}{x^2}} = 1/2.$$

(c)

$$\lim_{x\to\infty} \frac{2x^2 - 1}{4x^3 - 5x - 1} = \lim_{x\to\infty} \frac{2 - \frac{1}{x^2}}{4x - \frac{5}{x} - \frac{1}{x^2}} = 0$$

(d) $e^{-3x} \cos x$

6. **Little-o Notation.** The following notation is not taught in this course, but it is essential for any engineer or computer scientist. We say that $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ (in this sense, f grows more slowly than g). The '=' used here is not a true equality, as many distinct functions can be $o(g(x))$. Confirm the following.You don't need to finish the whole problem but it's a good practice.

(a) $x = o(x^2)$ and $x^{3/2} + \sqrt{x} = o(x^2)$. $\lim_{x\to\infty} \frac{x}{x^2} = 0$ $\quad \lim_{x\to\infty} \frac{x^{3/2}+\sqrt{x}}{x^2} = 0$

(b) For any $\alpha, \beta > 0$, $x^\alpha = o(x^{\alpha+\beta})$. $\lim_{x\to\infty} \frac{x^\alpha}{x^{\alpha+\beta}} = 0$ since $\beta > 0$

(c) For any $a \geq 0$ and $c > 1$, $x^a = o(c^x)$. $\lim_{x\to\infty} \frac{x^a}{c^x} = 0$ (looks familiar).

(d) $\ln(x) = o(x)$. Use the previous part but it's a bit tricky! A useful identity that you will use a lot is $x = e^{\ln(x)}$. Make it a fun exercise to verify this identity but feel free to just use it for now in this problem

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{\ln(x)}{e^{\ln(x)}} \overset{\text{let } y=\ln(x)}{=} \lim_{y\to\infty} \frac{y}{e^y} = 0$$

(e) For any $\alpha > 0$, $\ln(x) = o(x^\alpha)$. (Use a clever change of variables.)

$$\lim_{x\to\infty} \frac{\ln(x)}{x^\alpha} = \lim_{x\to\infty} \frac{\ln(x)}{e^{\ln x^\alpha}} = \lim_{x\to\infty} \frac{\ln(x)}{(e^\alpha)^{\ln(x)}} = \lim_{y\to\infty} \frac{y}{(e^\alpha)^y} = 0$$

$\to$ since $e^\alpha > 1$.

(f) $2^x = o(3^x)$. $\lim_{x\to\infty} \frac{2^x}{3^x} = 0$ (why?).

(g) For any $c > d > 1$, $d^x = o(c^x)$. $\lim_{x\to\infty} \left(\frac{d}{c}\right)^x$ since $\frac{d}{c} < 1$ we have the limit $\to 0$.

(h) $\ln(\ln(x)) = o(\ln(x))$. $\lim_{x\to\infty} \frac{\ln(\ln(x))}{\ln(x)} = \lim_{y\to\infty} \frac{\ln(y)}{y} = 0$ by (d)

(i) $e^{\sqrt{\ln(x)}} = o(\sqrt{x})$.

$$\lim_{x\to\infty} \frac{e^{\sqrt{\ln(x)}}}{\sqrt{x}} = \lim_{x\to\infty} \frac{e^{(\ln(x))^{1/2}}}{x^{1/2}} = \lim \frac{e^{(\ln(x))^{1/2}}}{e^{\frac{1}{2}\ln(x)}} = \lim_{x\to\infty} e^{\sqrt{\ln(x)}\,(1 - 1/2 \ln(x))} = 0$$

$\to -\infty$

(j) For any $\alpha > 0$, $e^{\sqrt{\ln(x)}} = o(x^\alpha)$.

Use α instead of $1/2$

(k) $\ln(x) = o(e^{\sqrt{\ln(x)}})$

$$\lim_{x\to\infty} \frac{\ln(x)}{e^{\sqrt{\ln(x)}}} \overset{\text{let } y=\ln(x)}{=} \lim_{y\to\infty} \frac{y}{e^{\sqrt{y}}} \overset{\text{let } z=\sqrt{y}}{=} \lim_{z\to0} \frac{z^2}{e^z}$$

we will do stuff like this a lot in the future. Changing var. is a good skill to have in math.

* (l) $c^x = o(x^x)$.

$$0 \leq \lim_{x\to\infty} \frac{c^x}{x^x} = \lim_{x\to\infty} \left(\frac{c}{x}\right)^x \leq \lim_{x\to\infty} \left(\frac{c}{c+1}\right)^x = 0 \Rightarrow \text{limit} \to 0$$

Conclusions: logarithms are slower than polynomials, which are slower than exponentials. But there are still functions slower, faster, and in between.

Bigger Conclusion: Do more Math!!! More interesting math is ahead