Math 199 CD2: Limit Definition of Derivative

September 6, 2021

1. Find the derivative of each function using the limit definition. Verify with differentiation rules.

   (a) $f(x) = 3x^2 - 2x + 2$

   (b) $V(a) = \sqrt{2a + 3}$

   (c) $f(x) = \dfrac{1}{x}$

   (d) $f(x) = \dfrac{x^2 + 1}{x - 2}$

2. You will eventually learn the following rule of taking derivative:

$$(f(x)+g(x))' = f(x)' + g(x)'$$

Verify this rule with the limit definition

3. For a function $f(x)$, define $f^+(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$ and $f^-(x) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$. We call $f^+(x)$ the right-derivative of $f(x)$, and $f^-(x)$ the left-derivative of $f(x)$.

   (a) Why do these names make sense?

   (b) For $f(x) = |x|$, find $f^+(0)$ and $f^-(0)$.

   (c) Show $f^-(x) = \lim_{h\to 0^+} \frac{f(x)-f(x-h)}{h}$.

   (d) Graphically interpret f^+ and f^-.

   (e) Find $f^+(0)$ and $f^-(0)$ for $f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0 \end{cases}$. Interpret this graphically.

(f) If $f(x)$ is not continuous at a, what can you say about $f^+(a)$ and $f^-(a)$?

(g) If $f(x)$ is differentiable at a, what can you say about $f^+(a)$ and $f^-(a)$?

(h) Find a and b so that $f(x)$ is differentiable everywhere, where $f(x) = \begin{cases} ax+2 & x \leq 1 \\ x^2+b & x > 1 \end{cases}$.

4. Find the point(s) at which the tangent line to the parabola $y = ax^2+bx+c$ is horizontal. (Notice that the solution to this problem locates the "nose" of the parabola.)

5. [**Hard Problem**] Consider the function:

$$\begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

Determine whether f is differentiable at 0.